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Note

On elements in small cocircuits in minimally k-connected graphs and matroids

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Abstract

We give a lower bound on the number of edges meeting some vertex of degree k in terms of the total number of edges in a minimally k-connected graph. This lower bound is tight if k is two or three. The extremal graphs in the case that $k=2$ are characterized. We also give a lower bound on the number of elements meeting some 2-element cocircuit in terms of the total number of elements in a minimally 2-connected matroid. This lower bound is tight and the extremal matroids are characterized. © 2002 Elsevier Science B.V. All rights reserved.

1. Introduction

The body of results concerning minimally k-connected graphs is rich and varied. A comprehensive survey of these results is given by Mader [5]. Previous results on minimally k-connected graphs focus on showing that there are many vertices of degree k in such a graph. Hence, there are many edges meeting a vertex of degree k in a minimally k-connected graph. This observation leads naturally to our investigation here of how many such edges are there in a minimally k-connected graph?

We use $v(G)$ and $e(G)$ to denote the number of vertices and edges, respectively, of a graph G. A graph G is *minimally k-connected* if and only if G is k-connected, and for each edge e of G, the deletion $G\backslash e$ is not k-connected. Dirac [1] proved that there are at least $(v(G)+4)/3$ vertices of degree 2 in a minimally 2-connected graph. Halin [2] proved that there are at least $(2v(G)+6)/5$ vertices of degree 3 in a minimally

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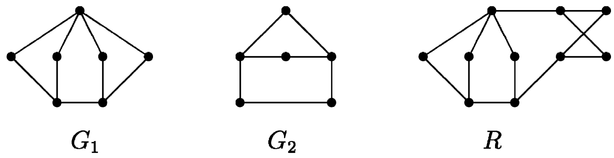


Fig. 1.

3-connected graph. Mader [4] extended these results by showing that there are at least $((k-1)v(G)+2k)/(2k-1)$ vertices of degree k in a minimally k-connected graph. The results of Dirac and Halin are best possible. The result of Mader for general k is very close to being best possible [4]. Theorems 1, 2, and 3 provide edge analogs of the results of Dirac, Halin, and Mader, respectively. They give lower bounds on the number of edges that meet a vertex of degree k in terms of the total number of edges in a minimally k-connected graph for $k \geqslant 2$. The statements of the main results of this paper are given in the remainder of this section. The proofs of these results are found in Sections 2 and 3.

Theorem 1 is the edge analog of Dirac's Theorem. The cycle graph on n vertices is denoted by C_n. The graphs G_1 and G_2 mentioned in Theorem 1 are given in Fig. 1. The graph R given in Fig. 1 is obtained from G_1 by replacing its rightmost vertex by a $K_{2,2}$. In general, the operation of replacing a vertex v of degree n in a graph G by a $K_{n,n}$ is described as follows. Vertex v is deleted from G and is replaced by two sets of new vertices A and B each containing n vertices. The subgraph induced by the vertices of A and B forms a $K_{n,n}$ with partite classes A and B. A matching is added between the n neighbors of the vertex v and the set A.

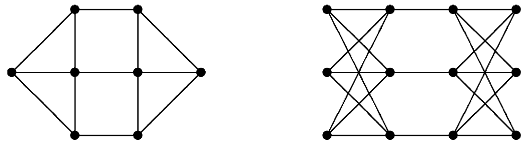
Theorem 1. *Let G be a minimally 2-connected graph with at least six edges. Then the number of edges of G meeting some vertex of degree two is at least $\lfloor (e(G)+7)/2 \rfloor$. Moreover, equality is attained in the previous bound if and only if either* (i) *G is isomorphic to G_1, G_2, C_6, C_7, or $K_{2,3}$, or* (ii) *G can be obtained from G_1, G_2, or $K_{2,3}$ by repeated application of the operation of replacing a vertex of degree two whose two neighbors have degree exceeding two by a $K_{2,2}$.*

The edge analog of Halin's result is given next.

Theorem 2. *Let G be a minimally* 3*-connected graph with at least* 14 *edges. The number of edges of G meeting some vertex of degree three is at least* $\lceil (2e(G)+12)/3 \rceil$.

The 13-edge graph given in Fig. 2(a) does not satisfy the conclusion of Theorem 2. An entire class of graphs attaining the lower bound in the above theorem is obtained starting from the graph $K_{3,4}$ by successively replacing a vertex of degree three by a $K_{3,3}$ at each stage [8, Theorem 4.5]. An example of such a graph is given in Fig. 2(b).

The edge analog of Mader's Theorem is given next.



(a) (b)

Fig. 2.

Theorem 3. *Let G be a minimally k-connected graph with at least $4k$ vertices, where $k \geqslant 3$. Then the number of edges of G meeting some vertex of degree k is at least $((k-1)e(G)+3k+1)/k$.*

Let G be a minimally k-connected graph. If $k=1$, then the conclusion of Theorem 3 holds for the graph G except when G is a path or a tree with exactly three vertices of degree one. If $k=2$ and $e(G) \geqslant 6$, then, by Theorem 1, the conclusion of Theorem 3 holds for the graph G except when (i) G is isomorphic to C_6 or $K_{2,3}$, or (ii) G can be obtained from $K_{2,3}$ by repeated application of the operation of replacing a vertex of degree two whose two neighbors have degree exceeding two by a $K_{2,2}$.

Mader's result implies that, for each minimally k-connected graph, $v_k(G)/v(G) \geqslant (k-1)/(2k-1)$, where $v_k(G)$ represents the number of vertices of degree k in G. He also gave a class of minimally k-connected graphs such that for each n-vertex member G_n of the class, $\lim_{n\to\infty} v_k(G_n)/v(G_n) = (k-1)/(2k-1)$. Likewise, Theorem 3 may be expressed as for each minimally k-connected graph G, $e_k(G)/e(G) \geqslant (k-1)/k$, where $e_k(G)$ represents the number of edges meeting a vertex of degree k. There does exist a class of minimally k-connected graphs such that for each n-vertex member G_n of the class, $\lim_{n\to\infty} e_k(G_n)/e(G_n) = (k-1)/k$. This class is obtained starting from $K_{k,k+1}$ by successively replacing vertices of degree k by copies of a complete bipartite graph with two classes of size k. Hence, the lower bound given in Theorem 3 is asymptotically best possible.

The lower bound in Theorem 3 may not be tight. The term of $3k+1$ that appears there could possibly be increased or the condition that $v(G) \geqslant 4k$ could possibly be decreased. However, the result remains asymptotically correct. We next describe a minimally k-connected graph G on $2k-1$ vertices that has $((k-1)e(G)+3k-1)/k$ edges meeting a vertex of degree k [5, Theorem 3]. Let H be the graph consisting of a set of $k-1$ isolated vertices and P_k be a disjoint path on k vertices. Let $G = P_k + H$. The set of vertices of degree k in G consists of the vertices of H together with the two end-vertices of P_k. All other vertices in the graph G have degree $k+1$.

The results of Dirac and Halin on minimally 2- and 3-connected graphs have been generalized to matroids by several authors including Lemos, Murty, Oxley [6,7,9], and Wong (see [11] for a comprehensive bibliography). In Theorem 4 we generalize our work on graphs to matroids in the case $k=2$ by computing a lower bound on the ratio of the number of elements meeting a cocircuit of size two to the total number of elements in a minimally 2-connected matroid.

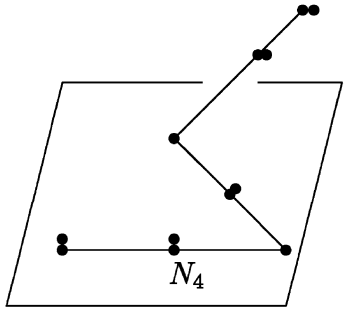


Fig. 3.

For each integer $t \geqslant 2$ we next describe a matroid N_t with rank t. If $t = 2$, then N_t has six elements and is formed by adding a point in parallel to each point of a three-point line. If $t \geqslant 2$, the matroid N_{t+1} is defined recursively by taking the two-sum (see [10, Section 7.1]) of the matroids N_2 and N_t. The matroid N_4 is pictured in Fig. 3 with all points but the join points pictured being in parallel classes of size two.

Theorem 4. *Let M be a minimally 2-connected matroid with corank at least two and at least six elements. Then the number of elements of M meeting a 2-element cocircuit is at least $\frac{2}{3}|E(M)| + 2$. Moreover, this bound is attained if and only if M is isomorphic to the dual of the matroid N_t, where t is the corank of M.*

The proof of Theorem 4 relies on a decomposition result for minimally 2-connected matroids of Oxley which is stated in Section 3. A similar generalization of Theorem 2 from graphs to matroids would be of interest. John Leo (personal communication) conjectured the following extension of Theorem 2.

Conjecture 5. Let M be a minimally 3-connected matroid with at least eight elements. Then the number of elements which meet a 3-element cocircuit is at least $(5|E(M)| + 30)/9$.

2. Minimally *k*-connected graphs

This section begins with some results and terminology used in the proofs of the graph results of the paper. The following fundamental result on minimal 3-connectivity in graphs of Halin [2, Satz 4] is used here.

Lemma 6. *Let e be an edge of a minimally 3-connected graph G and assume that e joins two vertices of degree exceeding three. Then G/e is minimally 3-connected.*

Throughout the remainder of the paper G is a minimally k-connected graph. The set of vertices of G of degree k is denoted by T and U is the set of vertices of G of degree at least $k+1$. Let $|T|=t$ and $|U|=u$. Let $e(T)$ and $e(U)$ be the number of edges of $G[T]$ and $G[U]$, respectively. Let $[T,U]$ denote the subset of edges of G such that for any such edge, one end-vertex is in T and the other end-vertex is in U. Let $e[T,U]$ denote the number of edges in $[T,U]$. Suppose that S is a subset of $V(G)$. We use $\omega(S)$ to denote the number of components of the subgraph induced by S. The number of edges of G meeting a vertex of degree k is denoted by $e_k(G)$ so that $e_k(G)=e[T]+e[T,U]$. The results of the following useful lemma are due to Mader. Part (a) of the lemma is from [3] and parts (b), (c), and (d) were proven in [4, Satz 3, Lemma 15, and Satz 2], respectively.

Lemma 7. *The following results hold*:

(a) $G[U]$ *is a forest*.
(b) $t \geqslant ((k-1)v(G)+2k)/(2k-1)$.
(c) *If* $\omega(U)<k$, *then* $e(T)+\omega(U)\geqslant k+1$, *when* $k\geqslant 2$.
(d) *If* $k\geqslant 3$, *then each vertex of* T *has at most* $k-2$ *neighbors in any one component of* U.

Several lemmas that are used in the proofs of Theorems 1–3 are given below.

Lemma 8. *The following results hold*:

(a) $e_k(G)=e(G)-u+\omega(U)$.
(b) $e[T,U]\geqslant (k-1)u+2\omega(U)$.

Proof. It follows from Lemma 7(a) that $e_k(G)=e(G)-e(U)=e(G)-(u-\omega(U))=e(G)-u+\omega(U)$. Hence part (a) of the lemma holds.

The sum of the degrees of the vertices of U is at least $(k+1)u$. Since $G[U]$ has $u-\omega(U)$ edges, $e[T,U]$ is at least $(k+1)u-2(u-\omega(U))=(k-1)u+2\omega(U)$. Hence part (b) of the lemma holds. $\square$

Proposition 9. $e_k(G)\geqslant (e(G)(k-1)+e(T)+\omega(U)(k+1))/k$

Proof. It follows from Lemma 8(a) and (b) that $e(G)-u+\omega(U)=e_k(G)=e[T,U]+e(T)\geqslant (k-1)u+2\omega(U)+e(T)$. Then $e(G)-u+\omega(U)\geqslant (k-1)u+2\omega(U)+e(T)$ implies that $u\leqslant (e(G)-\omega(U)-e(T))/k$. From substituting this upper bound for u into Lemma 8(a) we obtain that $e_k(G)\geqslant e(G)-(e(G)-\omega(U)-e(T))/k+\omega(U)=(e(G)(k-1)+e(T)+\omega(U)(k+1))/k$. $\square$

Proof of Theorem 1. Suppose that U is empty. Then every vertex of G has degree two. Then G is a cycle with at least six edges. Thus, $e_2(G)=e(G)\geqslant \lfloor (e(G)+7)/2\rfloor$. Moreover, equality holds in this equation if and only if G is isomorphic to C_6 or

C_7. Suppose that U is non-empty in the remainder of the proof. By [12, Corollary 3a], the forest $G[U]$ has at least two components. It follows from Proposition 9 that $e_2(G) \geqslant (e(G)+e(T)+3\omega(U))/2$. Hence, $\omega(U) \geqslant 2$ implies that $e_2(G) \geqslant (e(G)+6)/2$. Hence the lower bound of $\lfloor (e(G)+7)/2 \rfloor$ holds for $e_2(G)$.

It is straightforward to show that if G is isomorphic to G_1, G_2, $K_{2,3}$ or G can be obtained from one of these graphs by repeated application of replacing a vertex of degree two whose two neighbors have degree exceeding two by a $K_{2,2}$, then G has exactly $\lfloor (e(G)+7)/2 \rfloor$ edges that meet vertices of degree two.

Conversely, suppose that G has exactly $\lfloor (e(G)+7)/2 \rfloor$ edges that meet a vertex of degree two. We complete the proof by showing that the graph G is as given in the theorem statement. It follows from Proposition 9 and the fact that $\omega(U) \geqslant 2$ that $\omega(U)=2$. Assume that $e(G)$ is even. Then $e_2(G)=(e(G)+6)/2$. By Proposition 9, $e(T)=0$. Strict inequality in Lemma 8(b) would lead to strict inequality in Proposition 9. Thus, the equality in Proposition 9 implies equality in Lemma 8(b). Hence, $e[T,U]=u+4$. But $e(T)=0$ so that by summing the degrees of vertices in T we obtain that $e[T,U]=2t$. Thus, $2t=u+4=v(G)-t+4$. It follows that $3t=v(G)+4$. Thus, $v(G) \equiv 2 \pmod 3$. Hence, the number of vertices of degree two of G is $\lceil (v(G)+4)/3 \rceil$. It follows from [8, 2.15] that G is isomorphic to $K_{2,3}$ or G can be obtained from this graph by repeated application of the operation of replacing a vertex v of degree two by a $K_{2,2}$. The two neighbors of v must have degree exceeding two at each stage in order for the resulting graph to have exactly $\lfloor (e(G)+7)/2 \rfloor$ edges that meet a vertex of degree two.

Assume that $e(G)$ is odd. Then $e_2(G)=(e(G)+7)/2$. It follows from Proposition 9 that $e(T)$ is zero or one. By Lemma 8(a), $e(G)-u+\omega(U)=e_2(G)=(e(G)+7)/2$. Thus $u=(e(G)-3)/2$. Also, $e_2(G)=(e(G)+7)/2=(e(G)-3)/2+5=u+5$. Now we consider the two cases where $e(T)$ is zero or one. Suppose the former. Then $2t=e[T,U]=e_2(G)=u+5$. Hence, $2t=u+5=v(G)-t+5$. Therefore, $3t=v(G)+5$. It follows that $v(G) \equiv 1 \pmod 3$. Hence, the number of vertices of degree two of G is $\lceil (v(G)+4)/3 \rceil$. It follows from [8, 2.17] that G is isomorphic to G_1 or G can be obtained from this graph by repeated application of the operation of replacing a vertex of degree two by a $K_{2,2}$. Again, the neighbors of the vertex of degree two must both have degree exceeding two.

Suppose that $e(T)=1$. Then from summing degrees of the vertices of T we obtain that $2t-2=e[T,U]=e_2(G)-1=u+4=v(G)-t+4$. Thus $3t=v(G)+6$. It follows that $v(G) \equiv 0 \pmod 3$. Hence the number of vertices of G of degree two is $\lceil (v(G)+4)/3 \rceil$. It follows from [8, 2.19] that G is isomorphic to one of six base graphs given there or G can be obtained from one of these graphs by repeated application of the operation of replacing a vertex of degree two by a $K_{2,2}$. However, the only one of these six classes of graphs having an odd number of edges is the class obtained from the base graph K_3. Since G has at least six edges, G is not K_3. Upon replacing a vertex of degree two in K_3 by a $K_{2,2}$ we obtain the graph G_2. It now follows that G is obtained from the graph G_2 by the operation of replacing a vertex of degree two whose two neighbors have degree exceeding two by a $K_{2,2}$. $\square$

Proof of Theorem 2. Suppose the result does not hold. It follows from the case $k=3$ of Proposition 9 that $e(T)+4\omega(U) < 12$. Hence $\omega(U) < 3$. If $\omega(U)=0$, then $e_k(G)=e(G) \geqslant \lceil (2e(G)+12)/3 \rceil$ since $e(G) \geqslant 14$; a contradiction. Thus, $\omega(U)$ is one or two. Three useful observations about both of these cases are given in the next result.

Claim 10. (1) $u \neq \omega(U)$.
(2) *If* $u=\omega(U)+1$, *then* $e(G)=14$.
(3) *If some component of* U *has at least three vertices*, *then* $t \geqslant 8$.

Proof of Claim 10. Assume that $u=\omega(U)$. Then, by Lemma 8(a), $\frac{2}{3}e(G)+4 > e_3(G) = e(G)-u+\omega(U)=e(G)$. Thus $e(G) < 12$; a contradiction. Hence (1) holds. Suppose that $u=\omega(U)+1$. Then $\frac{2}{3}e(G)+4 > e_3(G)=e(G)-u+\omega(U)=e(G)-2+1=e(G)-1$. Thus $e(G) < 15$. Hence $e(G)=14$. Thus (2) holds. Suppose that some component S of U has at least three vertices. Let u_1 and u_2 be end-vertices of the tree S. Each vertex in T has at most one neighbor in S by Lemma 7(d). Thus, u_1 and u_2 together have at least six distinct neighbors in T. Some vertex of S other than u_1 and u_2 has at least two neighbors in T other than the six previously mentioned by the previous reasoning. Hence $t \geqslant 8$. Thus (3) holds. This completes the proof of Claim 10. $\square$

Suppose that $\omega(U)$ is one. Then $e(T)+4\omega(U) < 12$ implies that $e(T) \leqslant 7$. By Lemma 7(d), each vertex of T has at most one neighbor in U. Thus, each vertex of T has at least two neighbors in T. It follows from applying the Handshaking Lemma to $G[T]$ that $t \leqslant e(T) \leqslant 7$. It follows from Claim 10(1) that $\geqslant 2$. Since $e(T) \leqslant 7$, Claim 10(3) implies that the sole component of U has exactly two vertices. Then $u=\omega(U)+1$ and $e(G)=14$ by Claim 10(2). Let u_1 and u_2 be the two vertices of U. As before, these two vertices of u have at least six distinct neighbors in T. Thus t is six or seven. It follows from the Handshaking Lemma that $28=2e(G)=\deg(u_1)+\deg(u_2)+3t$. If $t=7$, then $\deg(u_1)+\deg(u_2)=7$. This implies that u_1 or u_2 has degree at most three; a contradiction. Hence $t=6$. Thus $\deg(u_1)+\deg(u_2)=10$. Hence, u_1 and u_2 have at least eight distinct neighbors in T. This contradicts that $t \leqslant 7$. Hence, $\omega(U)$ is two.

The fact that $e(T)+4\omega(U) < 12$ implies that $e(T) \leqslant 3$. By Lemma 7(d), each vertex of T has at least one neighbor in T. It follows from applying the Handshaking Lemma to $G[T]$ that $t/2 \leqslant e(T) \leqslant 3$. Hence $t \leqslant 6$. It follows from Claim 10(1) that $u \neq 2$. Suppose that $u=3$. Then Claim 10(2) implies that $e(G)=14$. The two components of $G[U]$ are an isolated vertex v_1 and an isolated edge (u_1,u_2). The vertices u_1 and u_2 have at least six distinct neighbors in T. Thus $t=6$. It follows from the Handshaking Lemma that $28=2e(G)=\deg(v_1)+\deg(u_1)+\deg(u_2)+3t$. Hence $\deg(v_1)+\deg(u_1)+\deg(u_2)=10$. Thus, some vertex of U has degree at most three; a contradiction.

We have established that $u \geqslant 4$. It follows from Claim 10(3) and the fact that $t \leqslant 6$ that $u=4$ and $G[U]$ consists of two isolated edges, say (u_1,u_2) and (v_1,v_2). As before,

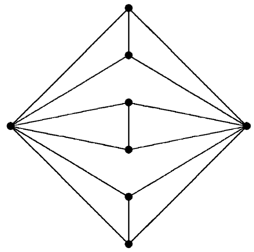


Fig. 4.

there are at least six neighbors of the vertices u_1 and u_2 in T. Thus $t=6$. Since $\frac{2}{3}e(G)+4 > e_3(G)=e(G)-u+\omega(U)=e(G)-2$, $e(G)<18$. Hence, the Handshaking Lemma implies that $34 \geqslant 2e(G)=\deg(u_1)+\deg(u_2)+\deg(v_1)+\deg(v_2)+3t \geqslant 4u+3t=34$. Thus, equality holds throughout the previous set of inequalities. Hence, each vertex of U has degree four and $e(G)=17$. Thus, each vertex of U has exactly three neighbors in T. From summing degrees of the vertices in T we obtain that $e_3(G)=3t-e(T)$. Thus $16 > \frac{2}{3}e(G)+4 > e_3(G)=3t-e(T)=18-e(T)$. Hence $e(T)\geqslant 3$. It follows that $e(T)=3$. Since each vertex of T is adjacent to at most one of u_1 and u_2 and is adjacent to at most one of v_1 and v_2, $G[T]$ consists of three isolated edges, (a_1,a_2), (b_1,b_2), and (c_1,c_2). Let H be the graph obtained by contracting edges (u_1,u_2) and (v_1,v_2). Then H is minimally 3-connected by Lemma 6. The graph H has 15 edges with exactly three edges contained in the subgraph of H induced by T. Hence, the graph H is as given in Fig. 4 with the six middle vertices being the vertices of T. However, this graph is not 3-connected; a contradiction. This completes the proof of Theorem 2. □

Proof of Theorem 3. The result follows from Proposition 9 if $e(T)+\omega(U)(k+1)\geqslant 3k+1$. For k being three, the result follows from Theorem 2. Hence, we may assume that $k\geqslant 4$ and $e(T)+\omega(U)(k+1)<3k+1$.

If each edge of G is incident to a vertex of degree k, then as G is k-connected with at least $4k$ vertices, $2e(G)\geqslant kv(G)\geqslant k\cdot 4k$. Thus $e(G)\geqslant 2k^2$. We conclude that $e_k(G)=e(G)\geqslant((k-1)e(G)+3k+1)/k$, noting that $e(G)\geqslant 2k^2\geqslant 3k+1$. Therefore, we assume that not every edge of G is incident with a vertex of degree k.

If $\omega(U)\geqslant k$, then $e(T)+\omega(U)(k+1)\geqslant k^2+k\geqslant 3k+1$; a contradiction. Thus $\omega(U)<k$. Then it follows from Lemma 7(c) that $3k\geqslant e(T)+\omega(U)(k+1)=e(T)+\omega(U)+k\omega(U)\geqslant k+1+k\omega(U)$. Thus $\omega(U)\leqslant 1$. Since not every edge of G is incident with a vertex of degree k, $\omega(U)\geqslant 1$. Thus $\omega(U)=1$. Hence, $3k\geqslant e(T)+\omega(U)(k+1)$ implies that $e(T)\leq 2k-1$. It follows from Lemma 7(d), each vertex of T has at least two neighbors in T. Again, by the Handshaking Lemma and Lemma 7(b), $2k=(4k^2-2k)/(2k-1)=((k-1)4k+2k)/(2k-1)\leqslant((k-1)v(G)+2k)/(2k-1)\leqslant t\leqslant e(T)\leqslant 2k-1$; a contradiction. This completes the proof of Theorem 3. □

3. Minimally 2-connected matroids

The proof of Theorem 4 is given in this section. We begin with some terminology which is specific to this section of the paper. A matroid M is *contraction-minimally connected* if and only if it is connected, but any matroid obtained from M by contracting a single element is not connected. The definition of the parallel connection and two-sum of two matroids may be found in [10, Chapter 7]. The following result is due to Oxley [6].

Lemma 11. *A matroid M is contraction-minimally connected if and only if it has at least three elements and either M is connected and has every element in a 2-circuit, or $M = P((M_1\backslash q_1; p_1),(M_2\backslash q_2; p_2))$ where both M_1 and M_2 are contraction-minimally connected matroids having at least five elements and rank at least two, and $\{p_1,q_1\}$ and $\{p_2,q_2\}$ are circuits of M_1 and M_2, respectively.*

Proof of Theorem 4. We prove the dual statement of Theorem 4. This statement is that in the contraction-minimally connected matroid M^*, $c_2(M^*) \geqslant \frac{2}{3}|E(M^*)| + 2$, where $c_2(M^*)$ is the number of elements that meet a 2-element circuit of M^*. Moreover, this bound is attained if and only if M^* is isomorphic to N_t. This proof is done by induction on $r^*(M)$.

Suppose that every element of M^* is in a 2-element circuit. Then $c_2(M^*) = |E(M^*)| \geqslant \frac{2}{3}|E(M^*)|+2$, since $|E(M^*)| \geqslant 6$. Assume that equality holds in the previous sentence. Then M^* has exactly six elements and rank at least two with every element in a non-trivial parallel class. It immediately follows that $M^* \cong N_2$. Thus, the result holds if every element of M^* is in a 2-element circuit. In particular, the result holds if $r^*(M) = 2$.

Suppose that $r^*(M)$ exceeds two, the result holds for matroids with rank r when $2 \leqslant r < r^*(M)$, and that not every element of M^* is in a 2-element circuit. Then, by Lemma 11, $M^* = P((M_1\backslash q_1; p_1),(M_2\backslash q_2; p_2))$ where both M_1 and M_2 are contraction-minimally connected matroids having at least five elements and rank at least two, and $\{p_1,q_1\}$ and $\{p_2,q_2\}$ are circuits of M_1 and M_2, respectively. Then

(a) $|E(M^*)| = |E(M_1)| + |E(M_2)| - 3$, and
(b) $c_2(M^*) \geqslant c_2(M_1) + c_2(M_2) - 4$,

with equality holding in (b) if and only if $\{p_1,q_1\}$ and $\{p_2,q_2\}$ are parallel classes of size two in M_1 and M_2, respectively. Thus $c_2(M^*) \geqslant c_2(M_1)+c_2(M_2)-4 \geqslant \frac{2}{3}|E(M_1)|+2+\frac{2}{3}|E(M_2)|+2-4 = \frac{2}{3}((|E(M_1)|+|E(M_2)|) = \frac{2}{3}(|E(M^*)|+3) = \frac{2}{3}|E(M^*)|+2$. Hence, the desired lower bound in this theorem holds. Suppose that equality holds in this bound. Hence equality holds in (b) and $\{p_1,q_1\}$ and $\{p_2,q_2\}$ are parallel classes of size two in M_1 and M_2, respectively. Either every element of $M_i, i\in\{1,2\}$ is in a 2-circuit or not. In the former case M_i is isomorphic to N_2 by previous comments. In the latter case $M_i \cong N_s$ for some $s \geqslant 3$ by the induction hypothesis. It is straightforward to check that the parallel connection of such matroids is isomorphic to N_t for some t. Thus $t = r^*(M)$. This completes the proof of Theorem 4. $\square$

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References

[1] G.A. Dirac, Minimally 2-connected graphs, J. Reine Angew. Math. 228 (1967) 204–216.
[2] R. Halin, Untersuchungen über minimale n-fach zusammenhängenden graphen, Math. Ann. 182 (1969) 175–188.
[3] W. Mader, Ecken vom Grad n in minimalen n-fach zusammenhängenden Graphen, Arch. Math. 23 (1972) 219–224.
[4] W. Mader, Zur Struktur minimal n-fach zusammenhängender graphen, Abh. Math. Sem. Univ. Hamburg 49 (1979) 49–69.
[5] W. Mader, On vertices of degree n in minimally n-connected graphs and digraphs, in: Combinatorics, Paul Erdös is Eighty, Vol. 2 (Keszthely, 1993), Bolyai Soc. Math. Stud., Vol. 2, János Bolyai Math. Soc., Budapest, 1996, pp. 423–449.
[6] J.G. Oxley, On connectivity in matroids and graphs, Trans. Amer. Math. Soc. 265 (1981) 47–58.
[7] J.G. Oxley, On matroid connectivity, Quart. J. Math. Oxford Ser. (2) 32 (126) (1981) 193–208.
[8] J.G. Oxley, On some extremal connectivity results for graphs and matroids, Discrete Math. 41 (1982) 181–198.
[9] J.G. Oxley, On minor-minimally-connected matroids, Discrete Math. 51 (1) (1984) 63–72.
[10] J.G. Oxley, Matroid Theory, Oxford University Press, New York, 1992.
[11] J.G. Oxley, Structure theory and connectivity for matroids, in: J. Bonin, J. Oxley, B. Servatius (Eds.), Matroid Theory: AMS-IMS-SIAM Joint Summer Research Conference on Matroid Theory, Contemporary Mathematics, Vol. 197, 1996, pp. 129–170.
[12] M.D. Plummer, On minimal blocks, Trans. Amer. Math. Soc. 134 (1968) 85–94.